

# Compactification of superstring theory

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## 1 Introduction

Superstring theories and M theory, at present the best candidate quantum theories which unify gravity, Yang-Mills fields and matter, are directly formulated in ten and eleven space-time dimensions. To obtain a candidate theory of our four dimensional universe, one must find a solution of one of these theories whose low energy physics is well described by a four dimensional effective field theory (EFT), containing the well established Standard Model of particle physics (SM) coupled to Einstein's general relativity. The standard paradigm for finding such solutions is compactification, along the lines originally proposed by Kaluza and Klein in the context of higher dimensional general relativity. One postulates that the underlying  $D$ -dimensional space-time is a product of four-dimensional Minkowski space-time, with a  $D - 4$ -dimensional compact and small Riemannian manifold  $K$ . One then finds that low energy physics effectively averages over  $K$ , leading to a four dimensional EFT whose field content and Lagrangian are determined in terms of the topology and geometry of  $K$ .

Of the huge body of prior work on this subject, the part most relevant for string/M theory is supergravity compactification, as in the limit of low energies, small curvatures and weak coupling, the various string theories and M theory reduce to ten and eleven dimensional supergravity theories. Many of the qualitative features of string/M theory compactification, and a good deal of what is known quantitatively, can be understood simply in terms of compactification of these field theories, with the addition of a few crucial ingredients from string/M theory. Thus, most of this article will restrict attention to this case, leaving many "stringy" topics to the articles on conformal field theory, topological string theory and so on. We also largely restrict attention to compactifications based on Ricci flat compact spaces. There is

an equally important class in which  $K$  has positive curvature; these lead to anti-de Sitter space-times and are discussed in the articles on AdS/CFT.

After a general review, we begin with compactification of the heterotic string on a three complex dimensional Calabi-Yau manifold. This was the first construction which led convincingly to the SM, and remains one of the most important examples. We then survey the various families of compactifications to higher dimensions, with an eye on the relations between these compactifications which follow from superstring duality. We then discuss some of the phenomena which arise in the regimes of large curvature and strong coupling. In the final section, we bring these ideas together in a survey of the various known four dimensional constructions.

## 2 General framework

Let us assume we are given a  $D = d + k$  dimensional field theory  $\mathcal{T}$ . A compactification is then a  $D$ -dimensional space-time which is topologically the product of a  $d$ -dimensional space-time with an  $k$ -dimensional manifold  $K$ , the compactification or “internal” manifold, carrying a Riemannian metric and with definite expectation values for all other fields in  $\mathcal{T}$ . These must solve the equations of motion, and preserve  $d$ -dimensional Poincaré invariance (or, perhaps another  $d$ -dimensional symmetry group).

The most general metric ansatz for a Poincaré invariant compactification is

$$G_{IJ} = \begin{pmatrix} f \eta_{\mu\nu} & 0 \\ 0 & G_{ij} \end{pmatrix},$$

where the tangent space indices are  $0 \leq I < d + k = D$ ,  $0 \leq \mu < d$ , and  $1 \leq i \leq k$ . Here  $\eta_{\mu\nu}$  is the Minkowski metric,  $G_{ij}$  is a metric on  $K$ , and  $f$  is a real valued function on  $K$  called the “warp factor.”

As the simplest example, consider pure  $D$ -dimensional general relativity. in this case, Einstein’s equations reduce to Ricci flatness of  $G_{IJ}$ . Given our metric ansatz, this requires  $f$  to be constant, and the metric  $G_{ij}$  on  $K$  to be Ricci flat. Thus, any  $K$  which admits such a metric, for example the  $k$  dimensional torus, will lead to a compactification.

Typically, if a manifold admits a Ricci flat metric, it will not be unique; rather there will be a moduli space of such metrics. Physically, one then expects

to find solutions in which the choice of Ricci flat metric is slowly varying in  $d$ -dimensional space-time. General arguments imply that such variations must be described by variations of  $d$ -dimensional fields, governed by an EFT. Given an explicit parameterization of the family of metrics, say  $G_{ij}(\phi^\alpha)$  for some parameters  $\phi^\alpha$ , in principle the EFT could be computed explicitly by promoting the parameters to  $d$ -dimensional fields, substituting this parameterization into the  $D$ -dimensional action, and expanding in powers of the  $d$ -dimensional derivatives. In pure GR, we would find the four-dimensional effective Lagrangian

$$\mathcal{L}_{EFT} = \int d^k y \sqrt{\det G(\phi)} R^{(4)} + \sqrt{\det G(\phi)} G^{ik}(\phi) G^{jl}(\phi) \frac{\partial G_{ij}}{\partial \phi^\alpha} \frac{\partial G_{kl}}{\partial \phi^\beta} \partial_\mu \phi^\alpha \partial_\mu \phi^\beta + \dots \quad (1)$$

While this is easily evaluated for  $K$  a symmetric space or torus, in general a direct computation of  $\mathcal{L}_{EFT}$  is impossible. This becomes especially clear when one learns that the Ricci flat metrics  $G_{ij}$  are not explicitly known for the examples of interest. Nevertheless, clever indirect methods have been found that give a great deal of information about  $\mathcal{L}_{EFT}$ ; this is much of the art of superstring compactification. However, in this section, let us ignore this point and continue as if we could do such computations explicitly.

Given a solution, one proceeds to consider its small perturbations, which satisfy the linearized equations of motion. If these include exponentially growing modes (often called “tachyons”), the solution is unstable.<sup>1</sup> The remaining perturbations can be divided into massless fields, corresponding to zero modes of the linearized equations of motion on  $K$ , and massive fields, the others. General results on eigenvalues of Laplacians imply that the masses of massive fields depend on the diameter of  $K$  as  $m \sim 1/\text{diam}(K)$ , so at energies far smaller than  $m$ , they cannot be excited.<sup>2</sup> Thus, the massive fields can be “integrated out,” to leave an EFT with a finite number of fields. In the classical approximation, this simply means solving their equations of motion in terms of the massless fields, and using these solutions to eliminate them from the action. At leading order in an expansion around a solution, these fields are zero and this step is trivial; nevertheless it is useful in making a systematic definition of the interaction terms in the EFT.

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<sup>1</sup>Note that this criterion is modified for AdS compactifications.

<sup>2</sup>This is not universal; given strong negative curvature on  $K$ , or a rapidly varying warp factor, one can have perturbations of small non-zero mass.

As we saw in pure GR, the configuration space parameterized by the massless fields in the EFT, is the moduli space of compactifications obtained by deforming the original solution. Thus, from a mathematical point of view, low energy EFT can be thought of as a sort of enhancement of the concept of moduli space, and a dictionary set up between mathematical and physical languages. To give its next entry, there is a natural physical metric on moduli space, defined by restriction from the metric on the configuration space of the theory  $\mathcal{T}$ ; this becomes the sigma model metric for the scalars in the EFT. Because the theories  $\mathcal{T}$  arising from string theory are geometrically natural, this metric is also natural from a mathematical point of view, and one often finds that much is already known about it. For example, the somewhat fearsome two derivative terms in Eq. (1), are (perhaps) less so when one realizes that this is an explicit expression for the Weil-Peterson metric on the moduli space of Ricci flat metrics. In any case, knowing this dictionary is essential for taking advantage of the literature.

Another important entry in this dictionary is that the automorphism group of a solution, translates into the gauge group in the EFT. This can be either continuous, leading to the gauge symmetry of Maxwell and Yang-Mills theories, or discrete, leading to discrete gauge symmetry. For example, if the metric on  $K$  has continuous isometry group  $G$ , the resulting EFT will have gauge symmetry  $G$ , as in the original example of Kaluza and Klein with  $K \cong S^1$  and  $G \cong U(1)$ . Mathematically, these phenomena of “enhanced symmetry” are often treated using the languages of equivariant theories (cohomology, K-theory, etc.), stacks, and so on.

To give another example, obstructed deformations (solutions of the linearized equations which do not correspond to elements of the tangent space of the true moduli space) correspond to scalar fields which, while massless, appear in the effective potential in a way which prevents giving them expectation values. Since the quadratic terms  $V''$  are masses, this dependence must be at cubic or higher order.

While the preceding concepts are general and apply to compactification of all local field theories, string and M theory add some particular ingredients to this general recipe. In the limits of small curvatures and weak coupling, string and M theory are well described by the ten and eleven dimensional supergravity theories, and thus the string/M theory discussion usually starts with Kaluza-Klein compactification of these theories, which we denote I, IIa,

IIB, HE, HO and M. Let us now discuss a particular example.

### 3 Calabi-Yau compactification of the heterotic string

Contact with the SM requires finding compactifications to  $d = 4$  either without supersymmetry, or with at most  $N = 1$  supersymmetry, because the SM includes chiral fermions, which are incompatible with  $N > 1$ . Let us start with the  $E_8 \times E_8$  heterotic string or “HE” theory. This choice is made rather than HO because only in this case can we find the SM fermion representations as subrepresentations of the adjoint of the gauge group.

Besides the metric, the other bosonic fields of the HE supergravity theory are a scalar  $\Phi$  called the dilaton, Yang-Mills gauge potentials for the group  $G \equiv E_8 \times E_8$ , and a two-form gauge potential  $B$  (often called the “Neveu-Schwarz” or “NS” two-form) whose defining characteristic is that it minimally couples to the heterotic string world-sheet. We will need their gauge field strengths below: for Yang-Mills, this is a two-form  $F_{IJ}^a$  with  $a$  indexing the adjoint of Lie  $G$ , and for the NS two-form this is a three-form  $H_{IJK}$ . Denoting the two Majorana-Weyl spinor representations of  $SO(1, 9)$  as  $S$  and  $C$ , then the fermions are the gravitino  $\psi_I \in S \otimes V$ , a spin 1/2 “dilatinos”  $\lambda \in C$ , and the adjoint gauginos  $\chi^a \in S$ . We use  $\Gamma_I$  to denote Dirac matrices contracted with a “zehnbein,” satisfying  $\{\Gamma_I, \Gamma_J\} = 2G_{IJ}$ , and  $\Gamma_{IJ} = \frac{1}{2}[\Gamma_I, \Gamma_J]$  etc.

A local supersymmetry transformation with parameter  $\epsilon$  is then

$$\delta\psi_I = D_I\epsilon + \frac{1}{8}H_{IJK}\Gamma^{JK}\epsilon \quad (2)$$

$$\delta\lambda = \partial_I\Phi\Gamma^I\epsilon - \frac{1}{12}H_{IJK}\Gamma^{IJK}\epsilon \quad (3)$$

$$\delta\chi^a = F_{IJ}^a\Gamma^{IJ}\epsilon. \quad (4)$$

We now assume  $N = 1$  supersymmetry. An unbroken supersymmetry is a spinor  $\epsilon$  for which the left hand side is zero, so we seek compactifications with a unique solution of these equations.

We first discuss the case  $H = 0$ . Setting  $\delta\psi_\mu$  in Eq. (2) to zero, we find that the warp factor  $f$  must be constant. The vanishing of  $\delta\psi_i$  requires

$\epsilon$  to be a covariantly constant spinor. For a six-dimensional  $M$  to have a unique such spinor, it must have  $SU(3)$  holonomy, in other words  $M$  must be a Calabi-Yau manifold. In the following we use basic facts about their geometry.

The vanishing of  $\delta\lambda$  then requires constant dilaton  $\Phi$ , while the vanishing of  $\delta\chi^a$  requires the gauge field strength  $F$  to solve the hermitian Yang-Mills equations,

$$F^{2,0} = F^{0,2} = F^{1,1} = 0.$$

By the theorem of Donaldson and Uhlenbeck-Yau, such solutions are in one-to-one correspondence with  $\mu$ -stable holomorphic vector bundles with structure group  $H$  contained in the complexification of  $G$ . Choose such a bundle  $E$ ; by the general discussion above, the commutant of  $H$  in  $G$  will be the automorphism group of the connection on  $E$  and thus the low energy gauge group of the resulting EFT. For example, since  $E_8$  has a maximal  $E_6 \times SU(3)$  subgroup, if  $E$  has structure group  $H = SL(3)$ , there is an embedding such that the unbroken gauge symmetry is  $E_6 \times E_8$ , realizing one of the standard grand unified groups  $E_6$  as a factor.

The choice of  $E$  is constrained by anomaly cancellation. This discussion (Green, Schwarz and Witten, 1987) modifies the Bianchi identity for  $H$  to

$$dH = \text{tr } R \wedge R - \frac{1}{30} \sum_a F^a \wedge F^a \quad (5)$$

where  $R$  is the matrix of curvature two-forms. The normalization of the  $F \wedge F$  term is such that if we take  $E \cong TK$  the holomorphic tangent bundle of  $K$ , with isomorphic connection, then using the embedding we just discussed, we obtain a solution of Eq. (5) with  $H = 0$ .

Thus, we have a complete solution of the equations of motion. General arguments imply that supersymmetric Minkowski solutions are stable, so the small fluctuations consist of massless and massive fields. Let us now discuss a few of the massless fields. Since the EFT has  $N = 1$  supersymmetry, the massless scalars live in chiral multiplets, which are local coordinates on a complex Kähler manifold.

First, the moduli of Ricci-flat metrics on  $K$  will lead to massless scalar fields: the complex structure moduli, which are naturally complex, and Kähler moduli, which are not. However, in string compactification the latter are complexified to the periods of the two-form  $B + iJ$  integrated over a basis of

$H_2(K, \mathbb{Z})$ , where  $J$  is the Kähler form and  $B$  is the NS two-form. In addition, there is a complex field pairing the dilaton (actually,  $\exp -\Phi$ ) and the “model-independent axion,” the scalar dual in  $d = 4$  to the two-form  $B_{\mu\nu}$ . Finally, each complex modulus of the holomorphic bundle  $E$  will lead to a chiral multiplet. Thus, the total number of massless uncharged chiral multiplets is  $1 + h^{1,1}(K) + h^{2,1}(K) + \dim H^1(K, \text{End}(E))$ .

Massless charged matter will arise from zero modes of the gauge field and its supersymmetric partner spinor  $\chi^a$ . It is slightly easier to discuss the spinor, and then appeal to supersymmetry to get the bosons. Decomposing the spinors of  $SO(6)$  under  $SU(3)$ , one obtains  $(0, p)$  forms, and the Dirac equation becomes the condition that these forms are harmonic. By the Hodge theorem, these are in one-to-one correspondence with classes in Dolbeault cohomology  $H^{0,p}(K, V)$ , for some bundle  $V$ . The bundle  $V$  is obtained by decomposing the spinor into representations of the holonomy group of  $E$ . For  $H = SU(3)$ , the decomposition of the adjoint under the embedding of  $SU(3) \times E_6$  in  $E_8$ ,

$$248 = (8, 1) + (1, 78) + (3, 27) + (\bar{3}, \bar{27}) \quad (6)$$

implies that charged matter will form “generations” in the 27, of number  $\dim H^{0,1}(K, E)$ , and “antigenerations” in the  $\bar{27}$ , of number  $\dim H^{0,1}(K, \bar{E}) = \dim H^{0,2}(K, E)$ . The difference in these numbers is determined by the Atiyah-Singer index theorem to be

$$N_{gen} \equiv N_{27} - N_{\bar{27}} = \frac{1}{2}c_3(E).$$

In the special case of  $E \cong TK$ , these numbers are separately determined to be  $N_{27} = b^{1,1}$  and  $N_{\bar{27}} = b^{2,1}$ , so their difference is  $\chi(K)/2$ , half the Euler number of  $K$ . In the real world, this number is  $N_{gen} = 3$ , and matching this under our assumptions so far is very constraining.

Substituting these zero modes into the ten-dimensional Yang-Mills action and integrating, one can derive the  $d = 4$  EFT. For example, the cubic terms in the superpotential, usually called Yukawa couplings after the corresponding fermion-boson interactions in the component Lagrangian, are obtained from the cubic product of zero modes

$$\int_K \Omega \wedge \text{Tr} (\phi_1 \wedge \phi_2 \wedge \phi_3),$$

where  $\Omega$  is the holomorphic  $\phi_i \in H^{0,1}(K, \text{Rep } E)$  are the zero modes, and  $\text{Tr}$  arises from decomposing the  $E_8$  cubic group invariant.

Note the very important fact that this expression only depends on the cohomology classes of the  $\phi_i$  (and  $\Omega$ ). This means the Yukawa couplings can be computed without finding the explicit harmonic representatives, which is not possible (we don't even know the explicit metric). More generally, one expects to be able to explicitly compute the superpotential and all other holomorphic quantities in the effective Lagrangian solely from "topological" information (the Dolbeault cohomology ring, and its generalizations within topological string theory). On the other hand, computing the Kähler metric in an  $N = 1$  EFT is usually out of reach as it would require having explicit normalized zero modes. Most results for this metric come from considering closely related compactifications with extended supersymmetry, and arguing that the breaking to  $N = 1$  supersymmetry makes small corrections to this.

There are several generalizations of this construction. First, the necessary condition to solve Eq. (5) is that the left hand side be exact, which requires

$$c_2(E) = c_2(TK). \quad (7)$$

This allows for a wide variety of  $E$  to be used, so that  $N_{gen} = 3$  can be attained with many more  $K$ . This class of models is often called "(0,2) compactification" to denote the world-sheet supersymmetry of the heterotic string in these backgrounds. One can also use bundles with larger structure group, for example  $H = SL(4)$  leads to unbroken  $SO(10) \times E_8$ , and  $H = SL(5)$  leads to unbroken  $SU(5) \times E_8$ .

The subsequent breaking of the grand unified group to the Standard Model gauge group is typically done by choosing  $K$  with non-trivial  $\pi_1$ , so that it admits a flat line bundle  $W$  with non-trivial holonomy (usually called a "Wilson line"). One then uses the bundle  $E \otimes W$  in the above discussion, to obtain the commutant of  $H \otimes W$  as gauge group. For example, if  $\pi_1(K) \cong \mathbb{Z}_5$ , one can use  $W$  whose holonomy is an element of order 5 in  $SU(5)$ , to obtain as commutant the SM gauge group  $SU(3) \times SU(2) \times U(1)$ .

Another generalization is to take the three-form  $H \neq 0$ . This discussion begins by noting that for supersymmetry, we still require the existence of a unique spinor  $\epsilon$ , however it will no longer be covariantly constant in the Levi-Civita connection. One way to structure the problem is to note that the right hand side of Eq. (2) takes the form of a connection with torsion;

the resulting equations have been discussed mathematically in (Li and Yau, 2004).

Another recent approach to these compactifications (Gauntlett, 2004) starts out by arguing that  $\epsilon$  cannot vanish on  $K$ , so it defines a weak  $SU(3)$  structure, a local reduction of the structure group of  $TK$  to  $SU(3)$  which need not be integrable. This structure must be present in all  $N = 1$ ,  $d = 4$  supersymmetric compactifications and there are hopes that it will lead to a useful classification of the possible local structures and corresponding PDE's on  $K$ .

## 4 Higher dimensional and extended supersymmetric compactifications

While there are similar quasi-realistic constructions which start from the other string theories and M theory, before we discuss these, let us give an overview of compactifications with  $N \geq 2$  supersymmetry in four dimensions, and in higher dimensions. These are simpler analog models which can be understood in more depth, and their study led to one of the most important discoveries in string/M theory, the theory of superstring duality.

As before, we require a covariantly constant spinor. For Ricci flat  $K$  with other background fields zero, this requires the holonomy of  $K$  to be one of trivial,  $SU(n)$ ,  $Sp(n)$ , or the exceptional holonomies  $G_2$  or  $Spin(7)$ . In table 1 we tabulate the possibilities with space-time dimension  $d$  greater or equal to 3, listing the supergravity theory, the holonomy type of  $K$ , and the type of the resulting EFT: dimension  $d$ , total number of real supersymmetry parameters  $N_s$ , and the number of spinor supercharges  $N$  (in  $d = 6$ , since left and right chirality Majorana spinors are inequivalent, there are two numbers).

The structure of the resulting supergravity EFT's is heavily constrained by  $N_s$ . We now discuss the various possibilities.

### 4.1 $N_s = 32$

Given the supersymmetry algebra, if such a supergravity exists, it is unique. Thus, toroidal compactifications of  $d = 11$  supergravity, IIA and IIB supergravity lead to the same series of maximally supersymmetric theories. Their

<i>theory</i>	<i>holonomy</i>	<i>d</i>	<i>N<sub>s</sub></i>	<i>N</i>
<i>M, II</i>	<i>torus</i>	any	32	max
<i>M</i>	<i>SU</i> (2)	7	16	1
	<i>SU</i> (3)	5	8	1
	<i>G</i> <sub>2</sub>	4	4	1
	<i>Sp</i> (4)	3	6	3
	<i>SU</i> (4)	3	4	2
	<i>Spin</i> (7)	3	2	1
	<i>SU</i> (2)	6	16	(1, 1)
<i>IIa</i>	<i>SU</i> (3)	4	8	2
	<i>G</i> <sub>2</sub>	3	4	2
<i>IIb</i>	<i>SU</i> (2)	6	16	(0, 2)
	<i>SU</i> (3)	4	8	2
	<i>G</i> <sub>2</sub>	3	4	2
<i>HE, HO, I</i>	<i>torus</i>	any	16	max/2
	<i>SU</i> (2)	6	8	1
	<i>SU</i> (3)	4	4	1
	<i>G</i> <sub>2</sub>	3	2	1

Table 1 – String/M theories, holonomy groups and the resulting supersymmetry

structure is governed by the exceptional Lie algebra  $E_{11-d}$ ; the gauge charges transform in a fundamental representation of this algebra, while the scalar fields parameterize a coset space  $G/H$  where  $G$  is the maximally split real form of the Lie group  $E_{11-d}$ , and  $H$  is a maximal compact subgroup of  $G$ . Nonperturbative duality symmetries lead to a further identification by a maximal discrete subgroup of  $G$ .

## 4.2 $N_S = 16$

This supergravity can be coupled to maximally supersymmetric super Yang-Mills theory, which given a choice of gauge group  $G$  is unique. Thus these theories (with zero cosmological constant and thus allowing super-Poincaré symmetry) are uniquely determined by the choice of  $G$ .

In  $d = 10$ , the choices  $E_8 \times E_8$  and  $Spin(32)/\mathbb{Z}_2$  which arise in string theory, are almost uniquely determined by the Green-Schwarz anomaly cancellation analysis. Compactification of these HE, HO and type I theories on  $T^n$  produces a unique theory with moduli space

$$SO(n, n+16; \mathbb{Z}) \backslash SO(n, n+16; \mathbb{R}) / SO(n, \mathbb{R}) \times SO(n+16, \mathbb{R}) \times \mathbb{R}^+. \quad (8)$$

In KK reduction, this arises from the choice of metric  $g_{ij}$ , the antisymmetric tensor  $B_{ij}$  and the choice of a flat  $E_8 \times E_8$  or  $Spin(32)/\mathbb{Z}_2$  connection on  $T^n$ , while a more unified description follows from the heterotic string world-sheet analysis. Here the group  $SO(n, n+16)$  is defined to preserve an even self-dual quadratic form  $\eta$  of signature  $(n, n+16)$ , for example  $\eta = (-E_8) \oplus (-E_8) \oplus I \oplus I \oplus I$  where  $I$  is the form of signature  $(1, 1)$  and  $E_8$  is the Cartan matrix. In fact, all such forms are equivalent under orthogonal integer similarity transformation, so the resulting EFT is unique. It has a rank  $16 + 2n$  gauge group, which at generic points in moduli space is  $U(1)^{16+2n}$ , but is enhanced to a non-abelian group  $G$  at special points. To describe  $G$ , we first note that a point  $p$  in moduli space determines an  $n$ -dimensional subspace  $V_p$  of  $\mathbb{R}^{16+2n}$ , and an orthogonal subspace  $V_p^\perp$  (of varying dimension). Lattice points of length squared  $-2$  contained in  $V_p^\perp$  then correspond to roots of the Lie algebra of  $G_p$ .

The other compactifications with  $N_S = 16$  is M theory on K3 and its further toroidal reductions, and IIB on K3. M theory compactification to  $d = 7$  is dual to heterotic on  $T^3$ , with the same moduli space and enhanced gauge

symmetry. As we discuss at the end of section 5, the extra massless gauge bosons of enhanced gauge symmetry are M2 branes wrapped on two-cycles with topology  $S^2$ . For such a cycle to have zero volume, the integral of the Kähler form and holomorphic two-form over the cycle must vanish; expressing this in a basis for  $H^2(K3, \mathbb{R})$  leads to exactly the same condition we discussed for enhanced gauge symmetry above. The final result is that all such K3 degenerations lead to one of the two dimensional canonical singularities, of types A, D or E, and the corresponding EFT phenomenon is the enhanced gauge symmetry of corresponding Dynkin type A, D, or E.

IIB on K3 is similar, but reducing the self-dual RR four-form potential on the two-cycles leads to self-dual tensor multiplets instead of Maxwell theory. The moduli space is Eq. (8) but with  $n = 5$ , not  $n = 4$ , incorporating periods of RR potentials and the  $SL(2, \mathbb{Z})$  duality symmetry of IIB theory.

One may ask if the  $Ns = 16$  I/HE/HO theories in  $d = 8$  and  $d = 9$  have similar duals. For  $d = 8$ , these are obtained by a pretty construction known as “F theory.” Geometrically, the simplest definition of F theory is to consider the special case of M theory on an elliptically fibered Calabi-Yau, in the limit that the Kähler modulus of the fiber becomes small. One check of this claim for  $d = 8$  is that the moduli space of elliptically fibered K3’s agrees with Eq. (8) with  $n = 2$ .

Another definition of F theory is the particular case of IIB compactification using Dirichlet seven-branes, and orientifold seven-planes. This construction is T-dual to the type I theory on  $T^2$ , which provides its simplest string theory definition. As discussed in (Polchinski, 1999), one can think of the open strings giving rise to type I gauge symmetry as living on 32 Dirichlet nine-branes (or D9-branes) and an orientifold nineplane. T-duality converts Dirichlet and orientifold  $p$ -branes to  $p - 1$ -branes; thus this relation follows by applying two T-dualities.

These compactifications can also be parameterized by elliptically fibered Calabi-Yaus, where  $K$  is the base, and the branes correspond to singularities of the fibration. The relation between these two definitions follows fairly simply from the duality between M theory on  $T^2$ , and IIB string on  $S^1$ . There is a partially understood generalization of this to  $d = 9$ .

Finally, these constructions admit further discrete choices, which break some of the gauge symmetry. The simplest to explain is in the toroidal com-

pactification of I/HE/HO. The moduli space of theories we discussed uses flat connections on the torus which are continuously connected to the trivial connection, but in general the moduli space of flat connections has other components. The simplest example is the moduli space of flat  $E_8 \times E_8$  connections on  $S^1$ , which has a second component in which the holonomy exchanges the two  $E_8$ 's. On  $T^3$ , there are connections for which the holonomies cannot be simultaneously diagonalized. This structure and the M theory dual of these choices is discussed in (de Boer et al, 2001).

### 4.3 $N_S = 8, d < 6$

Again, the gravity multiplet is uniquely determined, so the most basic classification is by the gauge group  $G$ . The full low energy EFT is determined by the matter content and action, and there are two types of matter multiplet.

First, vector multiplets contain the Yang-Mills fields, fermions and  $6 - d$  scalars; their action is determined by a prepotential which is a  $G$ -invariant function of the fields. Since the vector multiplets contain massless adjoint scalars, a generic vacuum in which these take non-zero distinct VEV's will have  $U(1)^r$  gauge symmetry, the commutant of  $G$  with a generic matrix (for  $d < 5$ , while there are several real scalars, the potential forces these to commute in a supersymmetric vacuum). Vacua with this type of gauge symmetry breaking, which does not reduce the rank of the gauge group, are usually referred to as on a "Coulomb branch" of the moduli space. To summarize, this sector can be specified by  $n_V$ , the number of vector multiplets, and the prepotential  $\mathcal{F}$ , a function of the  $n_V$  VEV's which is cubic in  $d = 5$ , and holomorphic in  $d = 4$ .

Hypermultiplets contain scalars which parameterize a quaternionic Kähler manifold, and partner fermions. Thus, this sector is specified by a  $4n_H$  real dimensional quaternionic Kähler manifold. The  $G$  action comes with triholomorphic moment maps; if nontrivial, VEVs in this sector can break gauge symmetry and reduce it in rank. Such vacua are usually referred to as on a "Higgs branch."

The basic example of these compactifications is M theory on a Calabi-Yau threefold. Reduction of the three-form leads to  $h^{1,1}(K)$  vector multiplets, whose scalar components are the CY Kähler moduli. The CY complex structure moduli pair with periods of the three-form to produce  $h^{2,1}(K)$  hyper-

multiplets. Enhanced gauge symmetry then appears when the  $CY_3$  contains ADE singularities fibered over a curve, from the same mechanism involving wrapped M2 branes we discussed under  $Ns = 16$ . If degenerating curves lead to other singularities (for example the ODP or “conifold”), it is possible to obtain extremal transitions which translate physically into Coulomb-Higgs transitions. Finally, singularities in which surfaces degenerate lead to non-trivial fixed point theories.

Reduction on  $S^1$  leads to IIA on  $CY_3$ , with the spectrum above plus a “universal hypermultiplet” which includes the dilaton. Perhaps the most interesting new feature is the presence of world-sheet instantons, which correct the metric on vector multiplet moduli space. This metric satisfies the restrictions of special geometry and thus can be derived from a prepotential.

The same theory can be obtained by compactification of IIB theory on the mirror  $CY_3$ . Now vector multiplets are related to the complex structure moduli space, while hypermultiplets are related to Kähler moduli space. In this case, the prepotential derived from variation of complex structure receives no instanton corrections, as we discuss in the next section.

Finally, one can compactify the heterotic string on  $K3 \times T^{6-d}$ , but this theory follows from toroidal reduction of the  $d = 6$  case we discuss next.

#### 4.4 $Ns = 8, d = 6$

These supergravities are similar to  $d < 6$ , but there is a new type of matter multiplet, the self-dual tensor (in  $d < 6$  this is dual to a vector multiplet). Since fermions in  $d = 6$  are chiral, there is an anomaly cancellation condition relating the numbers of the three types of multiplets (Aspinwall, 1996, section 6.6),

$$n_H - n_V + 29n_T = 273. \quad (9)$$

One class of examples is the heterotic string compactified on K3. In the original perturbative constructions, to satisfy Eq. (7), we need to choose a vector bundle with  $c_2(V) = \chi(K3) = 24$ . The resulting degrees of freedom are a single self-dual tensor multiplet and a rank 16 gauge group. More generally, one can introduce  $N_{5B}$  heterotic fivebranes, which generalize Eq. (7) to  $c_2(E) + N_{5B} = c_2(TK)$ . Since this brane carries a self-dual tensor multiplet, this series of models is parameterized by  $n_T$ . They are connected

by transitions in which an  $E_8$  instanton shrinks to zero size and becomes a fivebrane; the resulting decrease in the dimension of the moduli space of  $E_8$  bundles on K3 agrees with Eq. (9).

Another class of examples is F theory on an elliptically fibered Calabi-Yau threefold. These are related to M theory on an elliptically fibered CY<sub>3</sub> in the same general way we discussed under  $Ns = 16$ . The relation between F theory and the heterotic string on K3 can be seen by lifting M theory-heterotic duality; this suggests that the two constructions are dual only if the CY<sub>3</sub> is a K3 fibration as well. Since not all elliptically fibered CY<sub>3</sub>'s are K3 fibered, the F theory construction is more general.

We return to  $d = 4$  and  $Ns = 4$  in the final section. The cases of  $Ns < 4$  which exist in  $d \leq 3$  are far less studied.

## 5 Stringy and quantum corrections

The  $D$ -dimensional low energy effective supergravity actions on which we based our discussion so far are only approximations to the general story of string/M theory compactification. However, if Planck's constant is small,  $K$  is sufficiently large, and its curvature is small, they are controlled approximations.

In M theory, as in any theory of quantum gravity, corrections are controlled by the Planck scale parameter  $M_P^{D-2}$ , which sits in front of the Einstein term of the  $D$ -dimensional effective Lagrangian, and plays the role of  $\hbar$ . In general, this is different from the four dimensional Planck scale, which satisfies  $M_{Planck\ 4}^2 = \text{Vol}(K)M_P^{D-2}$ . After taking the low energy limit  $E \ll M_P$ , the remaining corrections are controlled by the dimensionless parameters  $l_P/R$  where  $R$  can any characteristic length scale of the solution: a curvature radius, the length of a non-trivial cycle, and so on.

In string theory, one usually thinks of the corrections as a double series expansion in  $g_s$ , the dimensionless (closed) string coupling constant, and  $\alpha'$ , the inverse string tension parameter, of dimensions (length)<sup>2</sup>. The ten dimensional Planck scale is related to these parameters as  $M_P^8 = 1/g_s^2(\alpha')^4$ , up to a constant factor which depends on conventions.

Besides perturbative corrections, which have power-like dependence on these

parameters, there can be world-sheet and “brane” instanton corrections. For example, a string-world sheet can wrap around a topologically non-trivial space-like two-cycle  $\Sigma$  in  $K$ , leading to an instanton correction to the effective action which is suppressed as  $\exp -\text{Vol}(\Sigma)/2\pi\alpha'$ . More generally, any  $p$ -brane wrapping a  $p$ -cycle can produce a similar effect. As for which terms in the effective Lagrangian receive corrections, this depends largely on the number and symmetries of the fermion zero modes on the instanton world-volumes.

Let us start by discussing some cases in which one can argue that these corrections are not present. First, extended supersymmetry can serve to eliminate many corrections. This is analogous to the familiar result that the superpotential in  $d = 4$ ,  $N = 1$  supersymmetric field theory does not receive (or “is protected from”) perturbative corrections, and in many cases follows from similar formal arguments. In particular, supersymmetry forbids corrections to the potential and two derivative terms in the the  $Ns = 32$  and  $Ns = 16$  lagrangians.

In  $Ns = 8$ , the superpotential is protected, but the two derivative terms can receive corrections. However, there is a simple argument which precludes many such corrections – since vector multiplet and hypermultiplet moduli spaces are decoupled, a correction whose control parameter sits in (say) a vector multiplet, cannot affect hypermultiplet moduli space. This fact allows for many exact computations in these theories.

As an example, in IIB on  $CY_3$ , the metric on vector multiplet moduli space is precisely Eq. (1) as obtained from supergravity, in other words the Weil-Peterson metric on complex structure moduli space. First, while in principle it could have been corrected by world-sheet instantons, since these depend on Kähler moduli which sit in hypermultiplets, it is not. The only other instantons with the requisite zero modes to modify this metric are wrapped Dirichlet branes. Since in IIB theory these wrap even dimensional cycles, they also depend on Kähler moduli and thus leave vector moduli space unaffected.

As previously discussed, for K3-fibered  $CY_3$ , this theory is dual to the heterotic string on  $K3 \times T^2$ . There, the vector multiplets arise from Wilson lines on  $T^2$ , and reduce to an adjoint multiplet of  $N = 2$  supersymmetric Yang-Mills theory. Of course, in the quantum theory, the metric on this moduli space receives instanton corrections. Thus, the duality allows deriving the exact moduli space metric, and many other results of the Seiberg-Witten theory of  $N = 2$  gauge theory, as aspects of the geometry of Calabi-Yau

moduli space.

In  $Ns = 4$ , only the superpotential is protected, and that only in perturbation theory; it can receive non-perturbative corrections. Indeed, it appears that this is fairly generic, suggesting that the effective potentials in these theories are often sufficiently complicated to exhibit the structure required for supersymmetry breaking and the other symmetry breakings of the SM. Understanding this is an active subject of research.

We now turn from corrections to novel physical phenomena which arise in these regimes. While this is too large a subject to survey here, one of the basic principles which governs this subject is the idea that string/M theory compactification on a singular manifold  $K$  is typically consistent, but has new light degrees of freedom in the EFT, not predicted by Kaluza-Klein arguments. We implicitly touched on one example of this in the discussion of M theory compactification on K3 above, as the space of Ricci-flat K3 metrics has degeneration limits in which curvatures grow without bound, while the volumes of two-cycles vanish. On the other hand, the structure of  $Ns = 16$  supersymmetry essentially forces the  $d = 7$  EFT in these limits to be non-singular. Its only noteworthy feature is that a non-abelian gauge symmetry is restored, and thus certain charged vector bosons and their superpartners become massless.

To see what is happening microscopically, we must consider an M theory membrane (or two-brane), wrapped on a degenerating two-cycle. This appears as a particle in  $d = 7$ , charged under the vector potential obtained by reduction of the  $D = 11$  three-form potential. The mass of this particle is the volume of the two-cycle multiplied by the membrane tension, so as this volume shrinks to zero, the particle becomes massless. Thus the physics is also well-defined in eleven dimensions, though not literally described by eleven dimensional supergravity.

This phenomenon has numerous generalizations. Their common point is that, since the essential physics involves new light degrees of freedom, they can be understood in terms of a lower dimensional quantum theory associated with the region around the singularity. Depending on the geometry of the singularity, this is sometimes a weakly coupled field theory, and sometimes a non-trivial conformal field theory. Occasionally, as in IIB on K3, the lightest wrapped brane is a string, leading to a “little string theory” (Aharony, 2000)

## 6 $N = 1$ supersymmetry in four dimensions

Having described the general framework, we conclude by discussing the various constructions which lead to  $N = 1$  supersymmetry. Besides the heterotic string on a  $CY_3$ , these compactifications include type IIA and IIB on orientifolds of Calabi-Yau threefolds, the related F theory on elliptically fibered Calabi-Yau fourfolds, and M theory on  $G_2$  manifolds. Let us briefly spell out their ingredients, the known non-perturbative corrections to the superpotential, and the duality relations between these constructions.

To start, we recap the heterotic string construction. We must specify a  $CY_3$   $K$ , and a bundle  $E$  over  $K$  which admits a hermitian Yang-Mills connection. The gauge group  $G$  is the commutant of the structure group of  $E$  in  $E_8 \times E_8$  or  $Spin(32)/\mathbb{Z}_2$ , while the chiral matter consists of metric moduli of  $K$ , and fields corresponding to a basis for the Dolbeault cohomology group  $H^{0,1}(K, \text{Rep } E)$  where  $\text{Rep } E$  is the bundle  $E$  embedded into an  $E_8$  bundle and decomposed into  $G$ -reps.

There is a general (though somewhat formal) expression for the superpotential,

$$W = \int \Omega \wedge + \text{Tr} \left( \bar{A} \bar{\partial} A + \frac{2}{3} \bar{A}^3 \right) + \int \Omega \wedge H^{(3)} + W_{NP}. \quad (10)$$

The first term is the holomorphic Chern-Simons action, whose variation enforces the  $F^{0,2} = 0$  condition. The second is the “flux superpotential,” while the third term is the non-perturbative corrections. The best understood of these arise from supersymmetric gauge theory sectors. In some but not all cases, these can be understood as arising from gauge theoretic instantons, which can be shown to be dual to heterotic five-branes wrapped on  $K$ . Heterotic world-sheet instantons can also contribute.

The HO theory is S-dual to the type I string, with the same gauge group, realized by open strings on Dirichlet nine-branes. This construction involves essentially the same data. The two classes of heterotic instantons are dual to D1 and D5-brane instantons, whose world-sheet theories are somewhat simpler.

If the  $CY_3$   $K$  has a fibration by tori, by applying T-duality to the fibers along the lines discussed for tori under  $Ns = 16$  above, one obtains various type II orientifold compactifications. On an elliptic fibration, double T-duality pro-

duces a IIB compactification with D7's and O7's. Using the relation between IIB theory on  $T^2$  and F theory on K3 fiberwise, one can also think of this as an F theory compactification on a K3-fibered Calabi-Yau fourfold. More generally, one can compactify F theory on any elliptically fibered fourfold to obtain  $N = 1$ . These theories have D3-instantons, the T-duals of both the type I D1 and D5-instantons.

The theory of mirror symmetry predicts that all  $CY_3$ 's have  $T^3$  fibration structures. Applying the corresponding triple T duality, one obtains a IIA compactification on the mirror  $CY_3$   $\tilde{K}$ , with D6-branes and O6-planes. Supersymmetry requires these to wrap special Lagrangian cycles in  $\tilde{K}$ . As in all Dirichlet brane constructions, enhanced gauge symmetry arises from coincident branes wrapping the same cycle, and only the classical groups are visible in perturbation theory. Exceptional gauge symmetry arises as a strong coupling phenomenon of the sort described in section 5. The superpotential can also be thought of as mirror to Eq. (10), but now the first term is the sum of a real Chern-Simons action on the special Lagrangian cycles, with disk world-sheet instanton corrections, as studied in open string mirror symmetry. The gauge theory instantons are now D2-branes.

Using the duality relation between the IIA string and eleven dimensional M theory, this construction can be lifted to a compactification of M theory on a seven dimensional manifold  $L$ , which is an  $S^1$  fibration over  $\tilde{K}$ . The D6 and O6 planes arise from singularities in the  $S^1$  fibration. Generically,  $L$  can be smooth, and the only candidate in table I for such an  $N = 1$  compactification is a manifold with  $G_2$  holonomy; therefore  $L$  must have such holonomy. Finally, both the IIA world-sheet instantons and the D2-instantons lift to membrane instantons in M theory.

This construction implicitly demonstrates the existence of a large number of  $G_2$  holonomy manifolds. Another way to arrive at these is to go back to the heterotic string on  $K$ , and apply the duality (again under  $Ns = 16$ ) between heterotic on  $T^3$  and M theory on K3 to the  $T^3$  fibration structure on  $K$ , to arrive at M theory on a K3-fibered manifold of  $G_2$  holonomy. Wrapping membranes on two-cycles in these fibers, we can see enhanced gauge symmetry in this picture fairly directly. It is an illuminating exercise to work through its dual realizations in all of these constructions.

Our final construction uses the interpretation of the strong coupling limit of the HE theory as M theory on a one-dimensional interval  $I$ , in which the

two  $E_8$  factors live on the two boundaries. Thus, our original starting point can also be interpreted as the heterotic string on  $K \times I$ . This construction is believed to be important physically as it allows generalizing a heterotic string tree-level relation between the gauge and gravitational couplings which is phenomenologically disfavored. One can relate it to a IIA orientifold as well, now with D8 and O8-branes.

These multiple relations are often referred to as the “web” of dualities. They lead to numerous relations between compactification manifolds, moduli spaces, superpotentials, and other properties of the EFT’s, whose full power has only begun to be appreciated.

## Further reading

Original references for all but the most recent of these topics can be found in the following textbooks and proceedings. We have also referenced a few research articles which are good starting points for the more recent literature. There are far more reviews than we could reference here, and a partial listing of these appears at <http://www.slac.stanford.edu/spires/reviews/>

*Superstring Theory*, M. B. Green, J. H. Schwarz and E. Witten, 2 vols, Cambridge University Press, 1987.

*Les Houches 1995: Quantum symmetries*, eds. A. Connes and K. Gawędzki, North Holland, 1998.

*String Theory*, J. Polchinski, 2 vols, Cambridge University Press, 1998.

*Quantum fields and strings: a course for mathematicians*, eds. P. Deligne et al., American Mathematical Society, 1999.

*Les Houches 2001: Unity from Duality: Gravity, Gauge theory and Strings*, eds. C. Bachas et al., Springer 2002.

*Strings and Geometry: proceedings of the 2002 Clay School*, eds. M. Douglas et al., American Mathematical Society, 2004.

O. Aharony, A brief review of ‘little string theories’, *Class.Quant.Grav.*17:929-938, 2000.

P. S. Aspinwall, K3 surfaces and string duality, 1996 preprint, arXiv:hep-th/9611137.

J. de Boer et al., Triples, fluxes, and strings, *Adv. Theor. Math. Phys.* **4**, 995 (2002).

J. Gauntlett, Branes, calibrations and supergravity, in *Strings and Geometry*, Douglas et al., pp. 79-126, AMS 2004.

J. Li and S.-T. Yau, The existence of supersymmetric string theory with torsion, 2004 preprint, arXiv:hep-th/0411136